Multifractal phenomenology and the refined similarity hypothesis in turbulence

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Within the framework of multifractal models of turbulence, regarding velocity increments and locally averaged energy dissipation together with their extension to the dissipation range, the following is shown. (1) If the one-dimensional surrogate for energy dissipation is correct, then the Kolmogorov-Oboukhov refined similarity hypothesis follows as a consistency condition between the two frameworks. (2) Validity of the one-dimensional surrogate is discussed under an added assumption that the higher-order moments of lateral and longitudinal derivatives have identical scaling. The two scaling exponents can be related in the form of an inequality for the full three-dimensional energy dissipation.

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The multifractal framework [1,2] is one of the general descriptions of intermittency in turbulence. In turbulence, two kinds of multifractality are expected. One is the $f(\alpha)-D(q)$ formalism [3,4] which deals with ϵ_r; energy dissipation locally averaged over scale r. The other, $d(h)-\zeta_p$, formalism [1] refers to (longitudinal) velocity difference $\delta u(r)$ between two points separated by distance r.

The refined similarity hypothesis [5] proposed by Kolmogorov and Oboukhov connects statistics of ϵ_r with that of $\delta u(r)$. This hypothesis has some support from numerical [6] and laboratory [7] experiments.

In this paper, we show how the refined similarity hypothesis follows as a consistency condition from the multifractality of turbulence with their extension to the dissipation range. We first employ the one-dimensional surrogate for energy dissipation. The case of the full three-dimensional dissipation is also discussed briefly.

We recapitulate the standard formalisms of multifractality in turbulence beginning with velocity increments [1]. For r in the inertial subrange, let $d(h)$ be the fractal dimension of the set where the velocity difference $\delta u(r)$ with separation r behaves as $\delta u(r) \propto r^h$. We assume that

$$\langle [\delta u(r)]^p \rangle \sim r^{\zeta_p},$$

where the angle brackets denote a spatial average and ζ_p is the scaling exponent.

For the energy dissipation averaged over size r, we assume that

$$\langle (\epsilon_r)^q \rangle \sim r^{(q-1)[D(q)-\Delta]},$$

where $D(q)$ is the generalized dimension and Δ is the spatial dimension (now $\Delta = 1$; it will be taken as 3 later).

To relate two formalisms we consider pth-order moments of pointwise energy dissipation. First we assume that the one-dimensional surrogate for energy dissipation

$$\epsilon(x) = \nu \left(\frac{\partial u}{\partial x}\right)^2 .$$

By extrapolating r down to a viscous scale (the so-called argument of intermediate dissipation range [8-10]), we have, in the $d(h)$ formalism [8],

$$\langle \epsilon(x)^p \rangle \sim \left\langle \left[\nu \left(\frac{\partial u}{\partial x}\right)^2\right]^p \right\rangle$$

$$\sim R_\lambda^{-6p+2[4p-\Delta+d(h)]/(1+h)}, \qquad (1)$$

where h is determined from

$$4p-\Delta = d'(h)(h+1) - d(h), \qquad (2)$$

and $R_\lambda \sim \nu^{-1/2}$ is the microscale Reynolds number. On the other hand, we have, in the $f(\alpha)$ formalism by a similar extrapolation [11],

$$\langle \epsilon(x)^p \rangle \sim \lim_{r\to 0} \langle (\epsilon_r)^p \rangle \sim R_\lambda^{6(q-p)}, \qquad (3)$$

where q is determined from

$$(q-1)(D(q)-\Delta) = 4(p-q). \qquad (4)$$

For the two kinds of multifractality to be consistent, Eqs. (1) and (3) should be equal. By equating the corresponding exponents, we obtain after a simplification

$$4p-\Delta = 3q(1+h) - d(h). \qquad (5)$$

Comparing with (2) we find $d'(h) = 3q$.

In general, $d(h)$ can be retrieved from ζ_p through the Legendre inverse transform

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$$d(h) = p\frac{d\zeta_p}{dp} - \zeta_p + \Delta, \qquad h = \frac{d\zeta_p}{dp}. \tag{6}$$

Setting $p = 3q$ in (6) and inserting the results into (5) we find

$$4p = 3q + \zeta_{3q} \tag{7}$$

and it follows from (4) that

$$(q-1)[D(q) - \Delta] = \zeta_{3q} - q.$$

Writing p for $3q$ we arrive at

$$\zeta_p = 1 + \left(\frac{p}{3} - 1\right) D\left(\frac{p}{3}\right),$$

for $\Delta = 1$. This is the well-known expression equivalent to

$$\epsilon_r \approx \frac{\delta u(r)^3}{r}, \tag{8}$$

where r is in the inertial subrange. Here $\approx$ implies that the moments of both sides at any positive order have the same scaling exponents [3,12,13]. Thus the refined similarity hypothesis in the *whole inertial range* follows as a consistency condition between the two frameworks of multifractal phenomenology and their extension to the dissipation range.

Next, we discuss the more general case of full three-dimensional energy dissipation by considering the validity of the one-dimensional surrogate. For simplicity we treat the case of the periodic boundary condition. It is necessary to relate statistics of longitudinal velocity differences with that of lateral ones. In addition to the isotropy assumption, here we assume for $p \geq 1$ that

$$\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^{2p} \right\rangle = A_p \left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^{2p} \right\rangle, \tag{9}$$

where A_p's are constants independent of the Reynolds number. This relation seems plausible, but (from isotropy and incompressibility) it can be derived rigorously only for the case of $p = 1$, $A_1 = 1/2$ [14].

By the definition

$$\epsilon(x) = \frac{\nu}{2} \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2$$

each term in $\langle \epsilon(x)^p \rangle$ in general has the form

$$\nu^p \left\langle \underbrace{\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} \cdots \frac{\partial u_m}{\partial x_n}}_{2p} \right\rangle, \tag{10}$$

where $i, j, ..., m, n$ are arbitrary indices.

Now, consider Hölder's inequality [15];

$$\langle |f_1 f_2 \cdots f_k|^q \rangle^{1/q} \leq \langle |f_1|^{p_1} \rangle^{1/p_1} \cdots \langle |f_k|^{p_k} \rangle^{1/p_k},$$

where

$$\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \leq 1.$$

Setting $q = 1$, $k = 2p$, and $p_1 = p_2 = \cdots = p_k = 2p$, we see that the term in angle brackets in (10) is bounded from above by

$$\left\langle \left(\frac{\partial u_i}{\partial x_j}\right)^{2p} \right\rangle^{1/2p} \left\langle \left(\frac{\partial u_k}{\partial x_l}\right)^{2p} \right\rangle^{1/2p} \cdots$$

$$\times \left\langle \left(\frac{\partial u_m}{\partial x_n}\right)^{2p} \right\rangle^{1/2p} \leq B_p \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^{2p} \right\rangle,$$

where the last line follows from isotropy and the assumption (9) and B_p's are constants independent of the Reynolds number. Therefore we have

$$\langle \epsilon(x)^p \rangle \leq \nu^p B_p \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^{2p} \right\rangle.$$

By the same argument as above we find in this case

$$\zeta_p \geq \frac{p}{3} + \left(\frac{p}{3} - 1\right)\left[D\left(\frac{p}{3}\right) - \Delta\right],$$

for $\Delta = 3$. Therefore for the general energy dissipation the two scaling exponents can be related in the form of an inequality with the assumption (9).

Needless to mention, the present remark neither substantiates multifractality nor the refined similarity hypothesis but it describes the relationship between them. In most laboratory experiments such as [3] the refined similarity hypothesis is assumed together with a one-dimensional surrogate for energy dissipation (and with Taylor's hypothesis). This note lends support for such a consistent interpretation of experimental data in terms of multifractality, although it seems difficult to obtain a solid experimental verification of multifractality.

At present there is no theoretical foundation for the velocity field to exhibit multifractality and the multifractal phenomenology of turbulence has even been challenged in [16]. In understanding intermittency of turbulence, it would be worthwhile to extract some information regarding the refined similarity hypothesis by analyzing the equations of motion.

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[1] G. Parisi and U. Frisch, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, Proceedings of the International School of Physics "Enrico Fermi," Course LXXXXVIII, Varenna, 1983, edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985), p. 84.

[2] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, and B.I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[3] C. Meneveau and K.R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).

[4] I. Hosokawa and K. Yamamoto, Phys. Fluids A **2**, 889 (1990).

[5] A.N. Kolmogorov, J. Fluid. Mech. **13**, 82 (1962); A.M. Oboukhov, *ibid.* **13**, 77 (1962).

[6] I. Hosokawa and K. Yamamoto, J. Phys. Soc. Jpn. **60**, 1852 (1991); and its addendum **62**, 380 (1993); S. Chen, G.D. Doolen, R.H. Kraichnan, and Z.-S. She, Phys. Fluids A **5**, 458 (1993).

[7] G. Stolovitzky, P. Kailasnath, and K.R. Sreenivasan, Phys. Rev. Lett. **69**, 1178 (1992); A.A. Praskovsky, Phys. Fluids A **4**, 2589 (1992); S.T. Thoroddsen and C.W. Van Atta, *ibid.* **4**, 2592 (1992).

[8] U. Frisch and M. Vergassola, Europhys. Lett. **14**, 439 (1991).

[9] M.H. Jensen, G. Paladin, and A. Vulpiani, Phys. Rev. Lett. **67**, 208 (1991).

[10] M. Nelkin, Phys. Rev. A **42**, 7226 (1990).

[11] M.S. Borgas, Philos. Trans. R. Soc. London, Ser. A **342**, 379 (1993). As pointed out by a referee, the extension of multifractality to the dissipation range (3) is equivalent to assume (8) at *the dissipation range.* But (3) does not imply (8) for general r in the inertial range.

[12] E.A. Aurell, U. Frisch, J. Lutsko, and M. Vergassola, J. Fluid Mech. **238**, 467 (1992).

[13] I. Hosokawa and K. Yamamoto, J. Phys. Soc. Jpn. **62**, 10 (1993).

[14] We note that $\partial u_1/\partial x_1$ is obtained from $\partial u_1/\partial x_2$ by an application of a linear operator whose Fourier representation is k_1/k_2, where (k_1, k_2, k_3) is wave number. Because the operator is not bounded, it does not follow from this kinematical relationship that they are comparable in L^p norms. The hypothesis (9), if correct, may involve some dynamics.

[15] This inequality holds both for the case of periodic boundary condition and for the unbounded space case with fluid at rest at infinity. In the latter case the homogeneity should be given up.

[16] I. Procaccia and P. Constantin, Phys. Rev. Lett. **70**, 3416 (1993).